CLARIFYING FUZZY LOGIC: A SPECTRAL DECOMPOSITION AND ICONIC REALIZATION *

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ABSTRACT

A model of the original form of fuzzy logic is presented. The fuzzy logic extension of the characteristic function of set theoretic sample points is reviewed as a lattice structure. The fuzzy logic treatment of the real unit interval as a chain with its ordering determined by the maximum and minimum of the lattice elements is discussed. The fuzzy membership function is a dimension function (a measure) on this chain.

A simple fuzzy logic is considered. Through its spectral decomposition, the model is shown to be realizable in terms of matrices which serve as "membership symbols." The membership function is shown to be a normalized trace---ie. the ratio of the trace of the membership symbol to the trace of the least upper bound membership symbol. The logical structure is decomposed into a direct sum/direct integral of pure idempotents. (In the continuum limit these are Dirac delta functions.) The characteristic function is completed as a vector space and the fuzzy membership symbol is embedded in it.

Simple icons are presented in order to visually exhibit some of the distinguishing features of the lattice operations. The matrix algebra is used to exemplify and compare differences among the non-complemented, totally ordered lattice of fuzzy logic; the orthocomplimented, distributive lattice of classical logic; and the non-distributive lattice of quantum logic. Operations are introduced which transform the classical orthocomplement and the fuzzy complement into eachother. Properties of the various representations are discussed.

INTRODUCTION

La logique est d'abord
une science naturelle.
------(Louis Rougièr, 1939)

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* This paper is dedicated to our friend and colleague H. Pierre Noyes who helped us to appreciate the empirical nature of scientific modeling .

This paper will present an overview of a simple fuzzy logical system as a non-complemented, totally ordered, and incomplete lattice. A spectral decomposition and iconic realizations of fuzzy logic will be developed in order to clarify some of the formal consequences of the fuzzy realization. Fuzzy logic will be contrasted with both classical and quantum logics. Finally we will review some of the interpretations of fuzzy logic in light of our comparisons.

Before considering the properties of fuzzy logic as a formal system, it will be convenient to give a brief summary of the basic formal postulates of lattices such as classical logic, fuzzy logic, and quantum logic. We will not go into a detailed mathematical analysis, but will provide a few simple examples to illustrate our basic points.

FORMAL FOUNDATIONS

Consider a set of elemental (atomic) propositions $A_i \in$ a set A ($i \in$ index set):

Lattice Postulates

Partial Ordering (figure 1a):

A partial ordering is denoted $\subseteq$ and read "contained within" or "implies." $A_i \subseteq A_j$ means that, for A_i, $A_j \in A$, whenever the criteria for attribute A_i is satisfied, then the criteria for attribute A_j is necessarily satisfied.

Three properties define a partial ordering:

P1 Reflexivity
If $A_i \in A$, then $A_i \subseteq A_i$.

P2 Antisymmetry
If A_i, $A_j \in A$ such that $A_i \subseteq A_j$ and $A_i \supseteq A_j$, then $A_i = A_j$. (Equality is defined a bidirectional antisymmetric relation.)

P3 Transitivity
If A_i, A_j, $A_k \in A$ such that $A_i \subseteq A_j$ and $A_j \subseteq A_k$, then $A_i \subseteq A_j$.

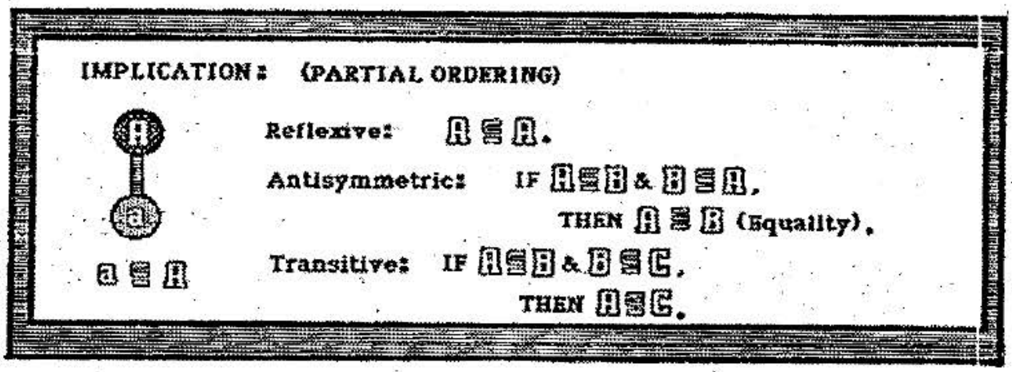


Fig. 1a: The Partial Ordering of a Lattice.

P4 <u>Adjunction</u> (figure 1b):
Adjunction (disjunction) is denoted (Ai ∪ Aj) and read "Ai or Aj."

> If Ai, Aj ∈ A, then there exists (Ai ∪ Aj) such that (Ai ∪ Aj) is the l.u.b. (Ai, Aj), the least upper bound of (Ai, Aj). The l.u.b. is the smallest attribute containing both Ai & Aj.

Ai ∪ Aj means that for Ai, Aj, ∈ A, whenever the criteria for either attribute Ai or for Aj is satisfied, then the criteria for Ai ∪ Aj is necessarily satisfied. For example, "person" is the l.u.b. ("boy," "girl"). (If Ai ⊆ "criteria" and Aj ⊆ "criteria," then Ai ∪ Aj ⊆ "criteria." For example, "boy" ⊆ "animal" and "girl" ⊆ "animal," then "person" ⊆ "animal.")

P5 <u>Conjunction</u> (figure 1c):
Conjunction is denoted (Ai ∩ Aj) and read "Ai and Aj."

> If Ai, Aj ∈ A, then there exists (Ai ∩ Aj) ∈ A such that (Ai ∩ Aj) is the g.l.b. (Ai, Aj), the greatest lower bound of Ai, Aj. The g.l.b. is the largest attribute contained by both Ai, Aj.

Ai ∩ Aj means that for Ai, Aj ∈ A, whenever the criteria for Ai and for Aj are simultaneously satisfied, then the criteria for Ai ∧ Aj are necessarily satisfied.

Postulates 1 - 5 define a <u>lattice</u>. [In the appendix we present several icons (Hasse diagrams) that illustrate the differing structures between some of the lattices which we will be considering within the body of the text.

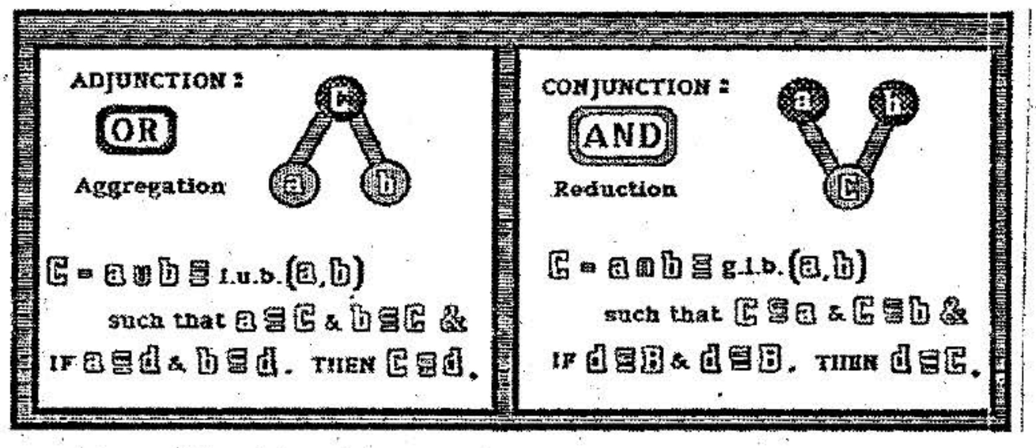


Fig. 1b & 1c: Adjunction and Conjunction on the lattice.

<u>Additional Properties</u>

P6 <u>Comparability</u>
If Ai, Aj ∈ A, and either Ai ⊃ Aj, Ai ⊂ Aj, or Ai = Aj, then the relation is a <u>total ordering</u>. The resulting lattice is called a <u>chain</u>. [Comparability does not generally apply to clas-

sical logic. We will refer to this property in our discussion of fuzzy logic, which as we shall see later, assumes comparability as part of its formal structure.]

P7 <u>Identity</u> (figure 1d)
There exists a proposition I such that $I \equiv$ l.u.b. (Ai! for all Ai $\in$ A). I is called the <u>trivial proposition</u> since it is <u>always</u> true. I is the least upper bound of the <u>entire</u> lattice.

P8 <u>Nullity</u> (figure 1e)
There exists a proposition Ø such that Ø $\equiv$ g.l.b. (Ai! for all Ai $\in$ A). Ø is called the <u>absurd proposition</u> since it is <u>always</u> false. Ø is the greatest lower bound of the <u>entire</u> lattice.

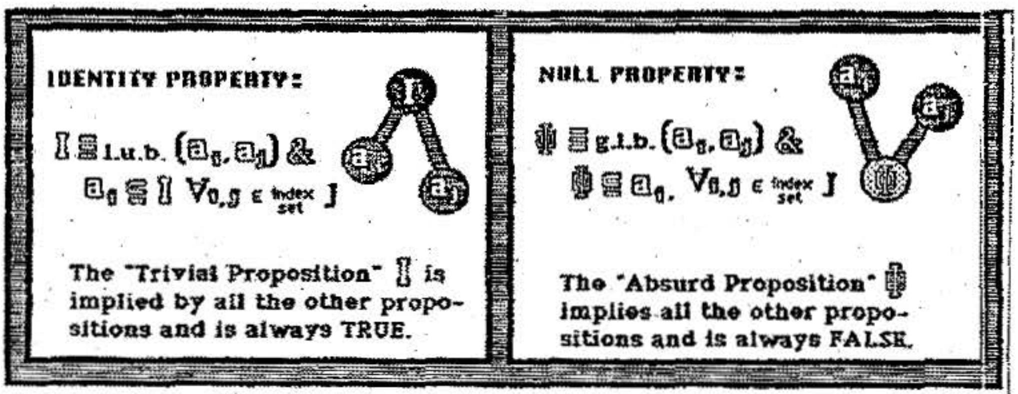


Fig. 1d & 1e: The Identity and Null Properties.

P9 <u>Complementation</u>
If Ai $\in$ A, then there exists $Ai^{\perp} \in$ A such that: (1) Ai $\cup$ $Ai^{\perp}$ = I and (2) Ai $\cap$ $Ai^{\perp}$ = Ø

P10 <u>Involution</u>
If Ai $\in$ A, then there exists $Ai^{\perp} \in$ A such that $(Ai^{\perp})^{\perp}$ = Ai.

Postulates P9 and P10 define an <u>orthocomplement</u>, an example of which is the traditional negation.

P11 <u>Distributivity</u>
a) If Ai, Aj, Ak $\in$ A, then there exists (Ai $\cap$ (Aj $\cup$ Ak)) $\in$ A and ((Ai $\cap$ Aj) $\cup$ (Ai $\cap$ Ak)) $\in$ A such that (Ai $\cap$ (Aj $\cup$ Ak)) = ((Ai $\cap$ Aj) $\cup$ (Ai $\cap$ Ak)).
b) If Ai, Aj, Ak $\in$ A, then there exists (Ai $\cup$ (Aj $\cap$ Ak)) $\in$ A and ((Ai $\cup$ Aj) $\cap$ (Ai $\cup$ Ak)) $\in$ A such that (Ai $\cup$ (Aj $\cap$ Ak)) = ((Ai $\cup$ Aj) $\cap$ (Ai $\cup$ Ak)).

The above propositions are for left distributivity. Similar propositions also hold for right distributivity.

P12 <u>Compatibility</u>
If Ai, Aj $\in$ A such that Ai = (Ai $\cap$ Aj) $\cup$ (Ai $\cap$ $Aj^{\perp}$) and (Ai $\cap$ Aj) $\cup$ ($Ai^{\perp}$ $\cap$ Aj) = Aj, then Ai and Aj are compatible. This is denoted Ai $\longleftrightarrow$ Aj.

P13 <u>Modularity</u>
If Ai, Aj, Ak, ∈ A, then Ai ∪ (Aj ∩ Ak) = Ai ∩ (Aj ∪ Ak). (I)

P14 <u>Weak Modularity</u>
If Ai, Aj ∈ A then for Ai ⊆ Aj, Ai ⟷ Aj.

P15 <u>Completeness</u>
A lattice, L, is called complete if for any sublattice Li ∈ L, there exists an l.u.b. and a g.l.b.

Venn Diagrams

Venn diagrams are a familiar pictorial representation of classical logic. In this approach one considers the relationship between points in a plane region, which is supposed to represent the sample space of some universe of discourse. One uses a technique of partitioning the sample space (usually by means of circles) into sets of sample points. This representation allows one to visualize which sets share points in common. This is illustrated in figure 2.

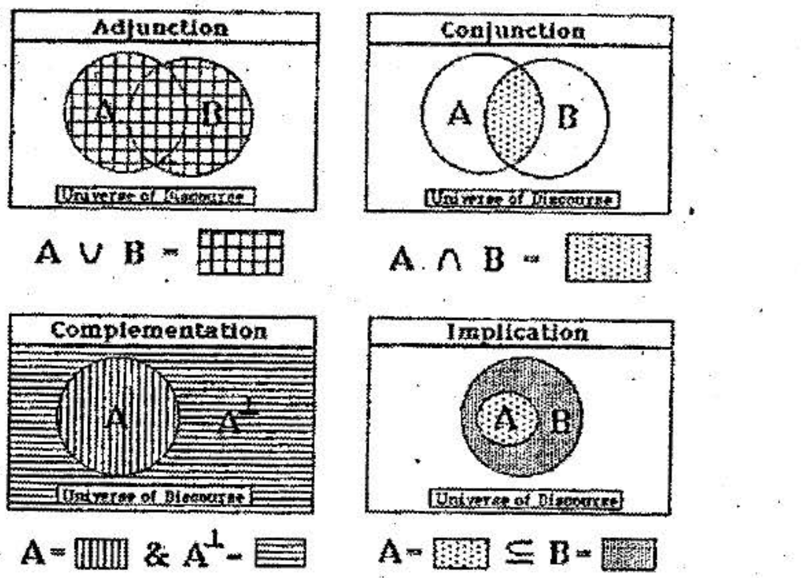


Fig.2: Venn Diagram representation of Classical Set Structures

A Point Set Isomorphism between Venn Diagrams and a Line Segment

We will now illustrate a one-to-one point-set identification between the Venn diagram realization of classical sets onto the set of points of a line segment. (fig. 3) [The Venn diagram topology is <u>not</u> preserved.] Observe that the point to point comparisons within each sample space allow us to identify the corresponding lattice structures. In this manner one can show that our identification preserves the lattice properties.

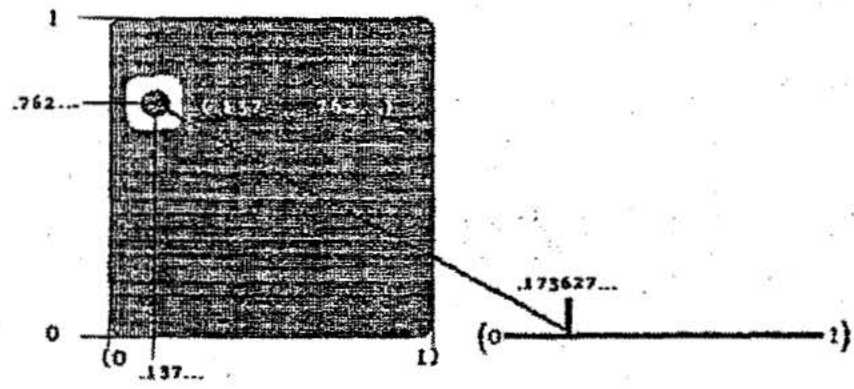


Fig. 3: An isomorphism between points in an "open" Universe of Discourse and points in the open unit interval. (The point set identification does not preserve the topology.)

INTRODUCING FUZZY LOGIC

In 1965, L. Zadeh introduced the mathematical theories of fuzzy sets and fuzzy logic (Zadeh, 1965). These fuzzy systems were developed in part as an attempt to represent linguistic, and thus, hopefully, cognitive qualifiers, called "hedges." In order to do this, Zadeh replaced the traditional Boolean "characteristic function," which has valuations in the set {0,1} (i.e. False, True), with a fuzzy membership function $\mu(A)$, which has valuations within the real interval. The boundary points of this fuzzy interval are then identified with the "classical" Boolean values. We will denote the characteristic function determined by the closed real unit interval [0,1] as the totally fuzzy membership function. And the characteristic function determined by the open real unit interval as the strictly fuzzy membership function.

Let us consider as examples:

1. "John likes Dick" vs "John likes Jane." These can be expressed as μ(Like/Dick; John) = 1, ie. "'John''s membership of 'Dick' in 'like' is 'True'." Likewise, μ(Like/Jane; John) = 1, which is read "'John''s membership of 'Jane' in 'like' is 'True'."

2. "John does not like Dick." It is denoted μ(not-Like/Dick; John) = 1, ie. "'John''s membership of 'Dick' in 'not-Like' is 'True'." This is identified with μ(Like/Dick; John) = 0, ie. "'John''s membership of 'Dick' in 'Like' is 'False'."

3. "John likes Dick very much" vs "John likes Bill somewhat." This might be denoted by, say, μ(Like/Dick; John) = .87 vs, say, μ(Like/Bill; John) = .63 in order to show that "'John' likes 'Dick' more than 'Bill'."

Zadeh goes on to argue that natural languages exhibit such gradations and that the fuzzy interpretation provides a way to realize the underlying mathematical structures.

Formal Fuzzy 2

(1) **Fuzzy Membership**

For a relation between a subject and an object, there exists a fuzzy membership function μ(relation/object;subject) which is meant to describe the extent of the relationship between them, μ:subject $\longrightarrow$ object, where μ(relation/object;subject) $\in$ (0,1) $\in$ R. For simplicity we will fix the subject and denote the qualification of the object by Ai [i $\in$ index set]. Thus μ(Ai) E (0,1).

(2) **Fuzzy Adjunction** [OR] - Fuzzy logic employs the max as a l.u.b. μ(Ai $\cup$ Aj) = max (μ(Ai), μ(Aj))

(3) **Fuzzy Conjunction** [AND] - Fuzzy logic employs the min as a g.l.b. μ(Ai $\cap$ Aj) = min (μ(Ai), μ(Aj))

These two fuzzy lattice connectives induce a <u>total ordering</u> (P6) on the lattice. This means that for all Ai, Aj $\in$ A , it is always the case that μ(Ai) $\subset$u(Aj), μ(Ai) $\supset$u(Aj), or μ(Ai) = μ(Aj). [In fig. 4 we illustrate a mapping from the set of point, which we identified with the set theoretic line segment of fig. 3, onto a directed line segment. The fuzzy lattice connectives induce a total ordering which makes the mapping mathematically feasible. We can always situate the set of "membership arrows" such that they have the same base point. We can then order them and "squeeze" any gaps out of a possibly discontinuous arrow by just renaming the points. This is not necessary but it makes the ordering structure more transparent.]

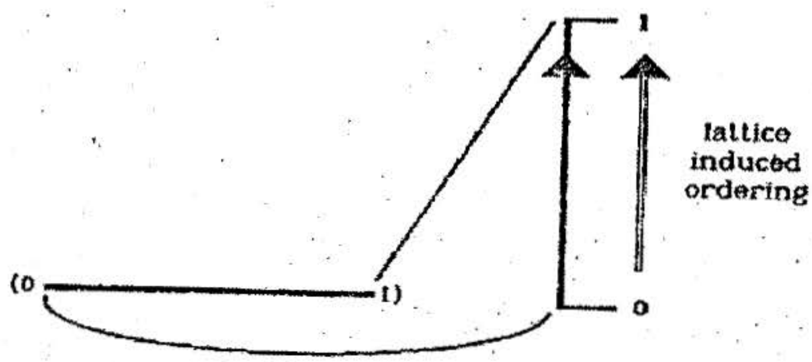


Fig. 4: Fuzzy logic's adjunction (max.) and conjunction (min.) induce a total ordering (min. $\leq$ max., always). It is traditional to depict the lattice ordering vertically with the least upper bound lying above the greatest lower bound. In the case of a chain such as fuzzy logic, the max. will lie above the min. This is called a Hasse diagram.

(4) **Fuzzy Complement**

Fuzzy logic does not have a traditional negative (ortho-complement) as defined by P9 and P10. Instead, it rejects the postulate of complementarity as defined by P9. An element of A, Ai^f, is defined such that $\mu(Ai^f) = 1 - \mu(Ai)$. Although we will use the term "**fuzzy complement**" when referring to the element Ai^f, it should be kept in mind that this is not a true complement.

In keeping with the "spirit of fuzzy," we will assume the strictly fuzzy domain, which does not include the identity, I, or the null set, Ø. As pointed out by Zadeh in his original development of fuzzy logic, the limit of the membership function, when valuated at the boundary points [Ø, I], is precisely the classical result. In a strictly fuzzy model the union of Ai with Ai^f does not give the entire universe of discourse [i.e. $\max(\mu(Ai), \mu(Ai^f)) < 1$ always], nor does the intersection of Ai with Ai^f yield the null set [i.e. $\min(\mu(Ai), \mu(Ai^f)) > 0$ always]. The fuzzy complement is still involutive, i.e. Ai^{ff} = Ai.

<u>Iconic Representation of Fuzzy Logic</u> (figure 5)

We will now present iconic realizations of fuzzy logic and observe that they have the same structure. Later in the paper we will present other visual tools for examining lattice structures. Our pictorial approach is presented in order to develop a clearer intuitive understanding of the structure of fuzzy logic.

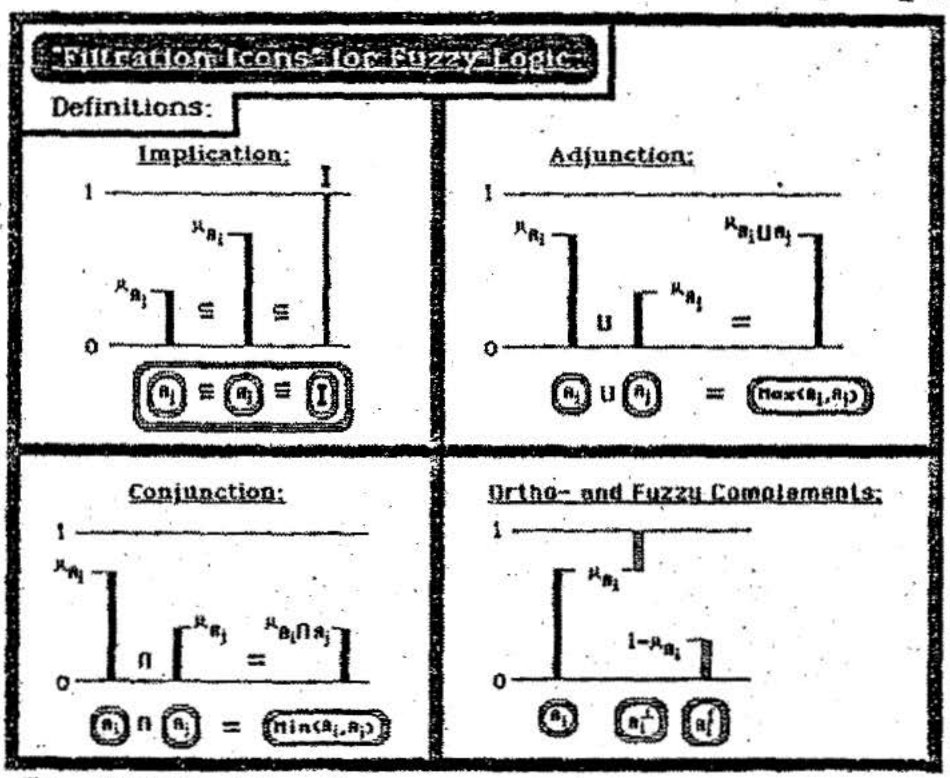


Fig. 5: "Filtration Icons" for Fuzzy Logic. (These visual tools depict the lattice structure of fuzzy logic.)

MATRIX REPRESENTATION OF FUZZY LOGIC

Fuzzy Membership Matrices:

In this section a matrix representation for fuzzy logic is provided. A matrix representation of a logic is a homomorphism (a many-to-one, structure preserving identification) from the lattice onto a collection of matrices such that the lattice properties are preserved. The sets of the propositional system will be identified with projection operators.

A projection operator is a matrix that is normal and idempotent. A matrix is normal if it commutes with its adjoint. Two operators A & B are said to commute if AB = BA, ie. if the order of multiplication does not make a difference. The adjoint of a matrix M is equal to its complex conjugate transposed. A matrix M is idempotent if $M^2 = M$. The eigenvalues of a projection operator are thus 0 & 1 on the real line.

A mapping will now be exhibited from the "filtration icons" of the previous section onto projection operators on the diagonal of a matrix. A norm will be defined on the space in order to provide for the concept of length. This norm will induce an inner product on the representation space. The inner product will allow us to discuss the concepts of orthogonality and disjointedness. Two subspaces A, B are orthogonal ($A \perp B$), if their inner product is zero. A and B are disjoint iff $A \subseteq B^{\perp}$. The matrix representations preserve the underlying lattice structure of fuzzy logic.

In our matrix representation the fuzzy membership operators become direct sums (direct integrals) of projection operators. "Every projection operator determines a subspace which is its range; and conversely, every subspace M determines a unique projection operator with range M." (Jauch, 1968a, p. 35.) The fuzzy membership function is recoverable as the trace of the normalized membership operators. The fuzzy membership function can be shown to provide a dimension function on the lattice. A dimension function (Birkhoff, 1961; Birkhoff and von Neumann, 1936; Jauch, 1968a, 1968b), v(a), must satisfy the following condition:

$$v(a) + v(b) = v(a \cup b) + v(a \cap b)$$

Clearly, the fuzzy membership function satisfies this requirement.

$$\mu(a) + \mu(b) = \mu(a \cup b) + \mu(a \cap b) = \max(\mu(a), \mu(b)) + \min(\mu(a), \mu(b))$$

The fuzzy space can be completed by taking the direct sum (direct integral) of a membership with its fuzzy complement. A complete inner product space is a Hilbert space. This allows us to take advantage of well known properties of Hilbert space and to compare fuzzy logic with the familiar matrix representations of classical and quantum logics. The membership matrices are seen to result from a spectral decomposition of fully compatible projection operators on the Hilbert space. This occurs because distributive lattices, such as fuzzy logic, which have commuting measurement operators, have fully reducible representations. [That the projection operators commute is equivalent to the subspaces being reduced. (Jordan, 1969; p. 37, esp. Thm. 12.7)] This is equivalent to the statement that all states are superselected or that the system is classical from a physical viewpoint. [Barut, 1971, p. 6; Beltrametti and Cassinelli, 1981; Jauch, 1968a, 1968b; Mackey, 1963; Piron, 1969; Wick, Wightman, and Wigner, 1952]. This is discussed further in the conclusion of the paper.

SPECTRAL DECOMPOSITION

Constructing Membership Matrices

A **spectral decomposition** of an operator provides a representation of the operator as composed by its projection operators. A **spectral measure** in Hilbert space is a measure over an operator's **spectral decomposition**. By the fundamental spectral threorem , there is a unique correspondence between self-adjoint operators and their spectral decomposition. "The one-to-one correspondence of the spectral measure on the real line to the self-adjoint operator [in a Hilbert space] permits us to replace one with the other." (Jauch, 1968, p. 35.) Thus, the spectral resolution of the operator is fully equivalent to its spectral decomposition. (Jauch, 1968a, 1972).

The filtration icons of the previous section can be alined on the diagonal of a matrix. We can consider the zero point of the fuzzy icon as beginning in the lower right hand corner of a unit matrix. The extension (length) of the fuzzy icon will equal the fuzzy membership and will collineate with the diagonal of the matrix. From this perspective, we see that fuzzy can be realized as a fully reducible direct sum or direct integral realization.

Let us define a membership operator M(Ai) in the following way

continuous: $M_{A_i} = \int_0^{\mu_{A_i}} |k\rangle dk \langle k|$ (.1a)

discrete: $M_{A_i} = \sum_{k=1}^{A_i} |k\rangle\langle k|$ (.1b)

(See figures 6a and 6b)

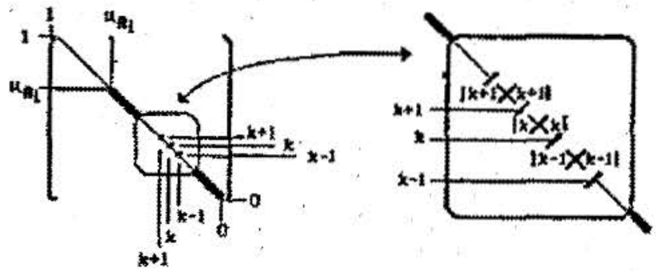
The trace is the sum of the diagonal elements of a matrix. A norm can be defined on our space as tr [M(A^t) M(A)]. Since M(A) is a self-adjoint (or Hermetian) operator, M(A) = M(A^t). [A self-adjoint operator has real spectra (or eigenvalues) when it is diagonalized.] The norm provides a homomorphism onto the fuzzy membership function. This gives us the concept of length addressed by fuzzy. We can add the additional angular information structure of orthogonality by having the norm induce an inner product upon the space (cf. pp. 244 - 245, Simmons), which we normalize thus tr [M(A) M(A)] / tr I.

Membership Symbols: [continuous]

Membership Matrices:

Definition:

$$\mathbb{M}_{A_i}^{(\text{Continuous})} = \int_0^{\mu_{A_i}} |k\rangle dk \langle k| = \int_0^1 |k\rangle \Theta(k-\mu_{A_i}) dk \langle k|$$



$$\mathbb{I}^{(\text{Continuous})} \equiv \int_0^1 |k\rangle dk \langle k| \equiv \int_0^1 d\,\mathbb{I}_{kk}$$

Properties:

- $\int_0^1 d\,\mathbb{I}_{ii}\, d\mathbb{I}_{jj} = \delta(i-j)\, d\mathbb{I}_{jj}$
- $\int_0^1 |i\rangle di \underbrace{\langle i|j\rangle}_{\delta(i-j)} = |j\rangle$
- $\mathrm{tr}\,\mathbb{I}^{(\text{Continuous})} = \mathrm{tr}\int_0^1 |k\rangle dk \langle k| = \int_0^1 \underbrace{\langle i|k\rangle}_{\delta(i-k)} di\, dk \langle k|i\rangle = \int_0^1 dk \underbrace{\langle k|k\rangle}_{1} = 1$

for fuzzy logic.
$\langle i|j\rangle = \delta(i-j)$
in quantum logic.
$\langle i|j\rangle \in$ (hyper-)complex numbers

Fig. 6a: Continuous Membership Symbols for Fuzzy Icons.
(The Θ Functional is defined in fig. 7a)

Since it is slightly more tractable mathematically, we will perform our calculations with the continuous case. The discrete case will follow essentially the same pattern. Notice that in the discrete case (fig. 6b) $|k\rangle\langle k|$ is the Ikk element of the identity. [One shows trivially that the membership symbols M(Ai) are idempotent, M(Ai)M(Ai) = M(Ai).]

Rather than integrating from 0 to μ(Ai), for which the limits are different each time we do the integration, it is convenient to use simplifying mathmatical methods which will allow us to

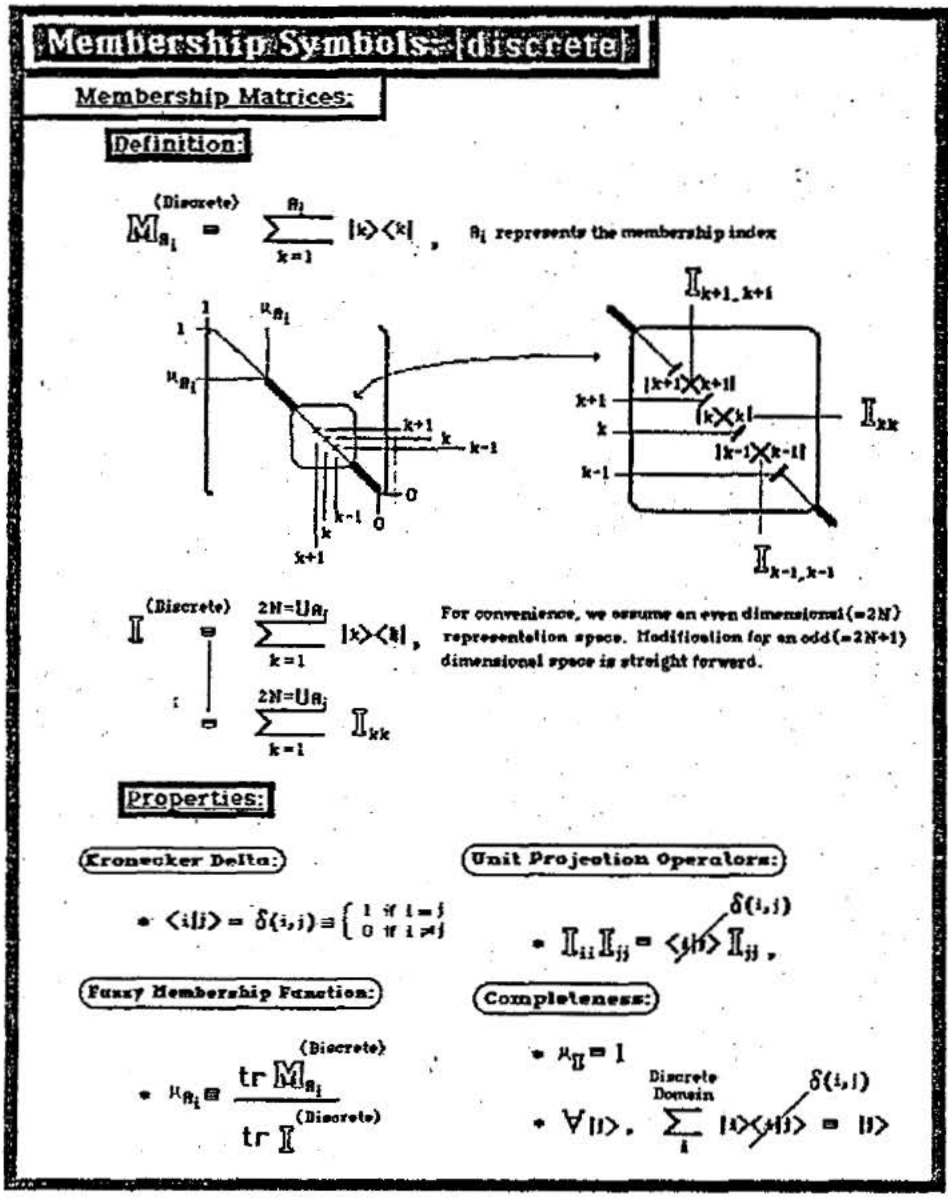


Fig. 6b: Discrete Membership Symbols for Fuzzy Icons.

integrate over the same domain [0 to 1] for all cases. In order to accomplish this, let us introduce the Heavyside step functional (see figure 7a).

$$\Theta(\mu(Ai) - k) = 0 \text{ if } k < \mu(Ai); \quad = 1 \text{ if } k > \mu(Ai) \qquad (.2)$$

or

$$\Theta(k - \mu(Ai)) = 1 - \Theta(\mu(Ai) - k) = 1 \text{ if } \mu(Ai) < k; \quad = 0 \text{ if } \mu(Ai) > k \qquad (.3)$$

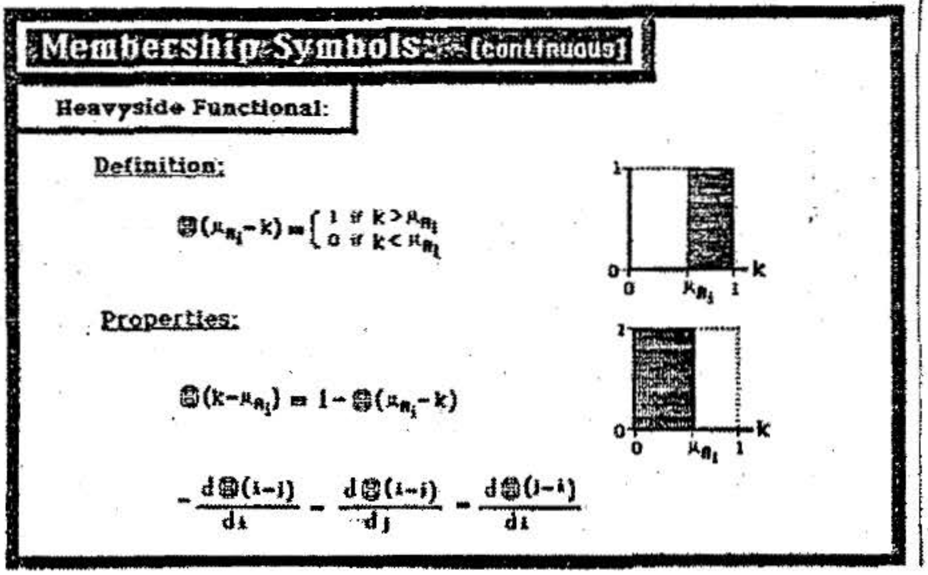


Fig. 7a: The Heavyside Step Functional.

Combining (.3) with (.1)

$$M(Ai) = \int_0^1 |k\rangle\, \Theta(k - \mu(A_i))\, dk \langle k| \qquad (.4)$$

The derivative of the Heavyside function is a Dirac delta functional (see figure 7b).

$$\frac{d\,\Theta(k-\mu(A_i))}{dk} = -\,\delta(k-\mu(A_i)) \qquad (.5)$$

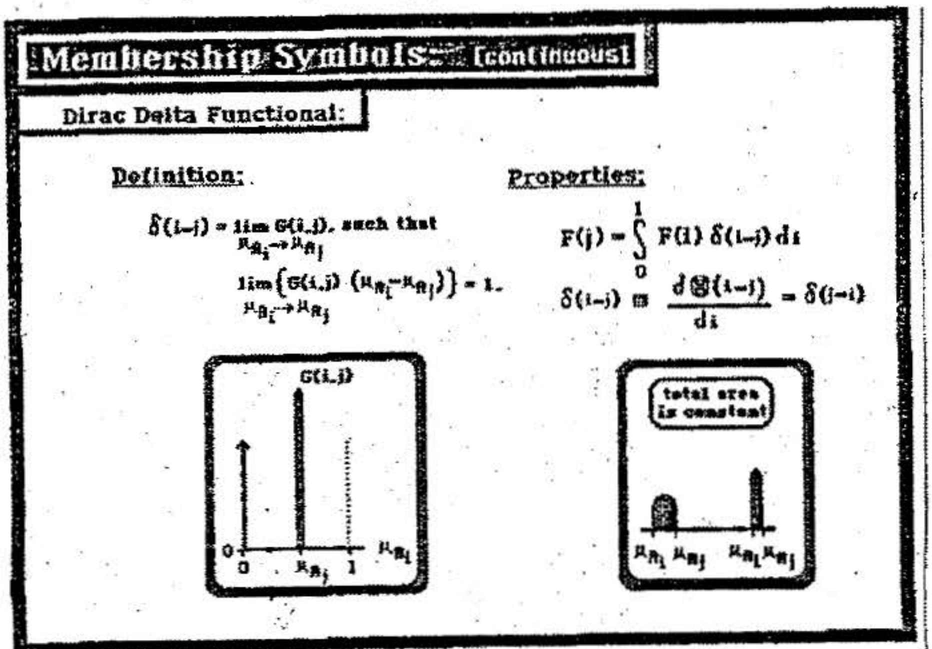


Fig. 7b: The Dirac Delta Functional

In the continuum case, one can define a <u>membership density</u>

$$dM(Ai)/d\mu(Ai) = -\,|\mu(Ai))(\mu(Ai)| \qquad (.6)$$

The negative of of this density is an idempotent projection operator. Each of these operators is orthogonal to all other such operators. The set of these densities provide an orthonormal basis for decomposing the unit.

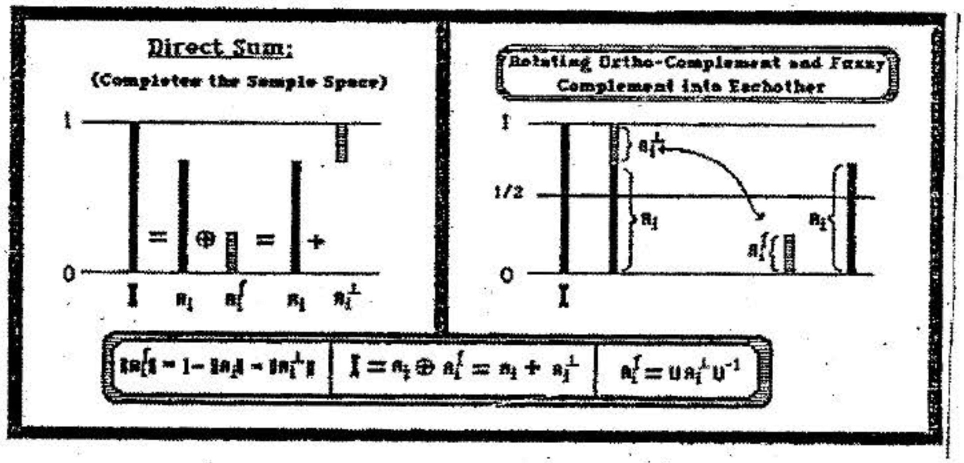
<u>Completing the Space</u>

The fuzzy lattice is <u>distributive</u>. If one takes the <u>trace</u> or sum of the diagonal elements of M(A), the membership operator, one obtains the fuzzy membership function. In the discrete case one must take a normalized trace, i.e. the ratio of the trace of the fuzzy membership matrix to the trace of the identity matrix.

The identity matrix is identified with the least upper bound of the domain which represents the total universe of discourse. One can reconstruct the identity by completing the space. We do this by taking the direct sum (or integral) of M(Ai) and $M(Ai^f)$. (see figure 8a)

$$M(Ai) \oplus M(Ai^f) = \mathbb{I} \qquad (.7a)$$

$$= M(Ai) + M(Ai^\perp) \qquad (.7b)$$



Figs. 8a & 8b: A Completed Fuzzy Space and the Relation between the Ortho-Complement and the Fuzzy Complement

Thus, we see that since it is strictly true that $\Phi \subset M(Ai)$, $M(Ai^f) \subset \mathbb{I}$, neither M(Ai) or $M(Ai^f)$ represents a subspace of the other. Also, both are proper subspaces of the (complete) Hilbert space. It is important to note, however, that M(Ai) and $M(Ai^f)$ are <u>not</u> orthogonal although they can be constructed from a complete orthonormal basis of projection operators (2).

Transforming the Fuzzy Complement and the Orthocomplement into Each Other

Let us call an operator A the conjugate of an operator B by a transformation operator T if $A = T\,B\,T^{-1}$. Because the trace is order independent ($\mathrm{tr}\,[A\,B] = \mathrm{tr}\,[B\,A]$), such a transformation is trace preserving ($\mathrm{tr}\,A = \mathrm{tr}\,[T\,B\,T^{-1}] = \mathrm{tr}\,[B\,T^{-1}\,T] = \mathrm{tr}\,B$). In a Hilbert space, a trace preserving transformation is a unitary (or antiunitary, by Wigner's Theorem) transformation.

We will now exhibit such a unitary transformation that maps the orthogonal complement and the fuzzy complement into each other, making them conjugates (see figures 8b & 9).

$$M(Ai^{f}) = UM(Ai^{\perp})U^{-1} \qquad (.8)$$

The trace of the normalized identity is 1.

$$\mathrm{tr}\,\mathbb{I} = 1 = \left\langle \left[\int_0^1 |k\rangle\, dk \,\langle k|\right]\right\rangle = \mathrm{tr}\,\mathbb{I}_{kk'}\mathbb{I}_{jj} \qquad (.9)$$

Since the trace is conserved under unitary transformations of the whole space.

$$\mathrm{tr}\; M(A^{f}) = \mathrm{tr}\; M(A^{\perp}) = 1 - \mu(A) \qquad (.10)$$

One can intuitively think of the projection operators of the spectral decomposition as slots along the diagonal of a matrix. (Orlov, 1975, 1978; Schwinger, 1970, who begins by literally "quantizing classical logic" within a spectral realization; Trainor and Wise, 1979, esp. sect. 5.3 where a "slot" approach to representations of symmetry groups is discussed.) We illustrate this in figs. 6a & 6b. In fig. 10 we illustrate the filtration process that occurs during the multiplication of "fuzzy slots."

DISCUSSION

Since the fuzzy lattice structure can be represented by simultaneously diagonalized matrices, it is a proper subclass of a more general matrix algebra which admits off-diagonal elements. 3
This occurs for non-distributive lattices such as are found in quantum logics and is a consequence of non-commuting projection operators. The non-commuting algebra of the observables of physical experience is constructions out of such operators. It is possible to operationally construct an empirical logic for empirically discernable alternatives or empirical questions (Birkhoff and von Neumann, 1936; Finkelstein, 1963, 1968a, 1968b; Jauch, 1968a, 1968b; Mackey, 1963; Oshins and McGoveran, 1979; Oshins, 1982; Piron 1964, 1976; Putnam, 1968; von Neumann, 1955).

Upon being exponentiated, the quantum commutators generate the symmetries and transformations of physical experience. They form natural equivalence class stuctures of alternative possible, although mutually exclusive, bases within the Hilbert space that carries the representations. We suspect that such irreducible equivalence class structures are important for abstract thought.

Ortho-to Fuzzy Complement

Given: $$\mathbb{M}_A \equiv \int_0^1 |i\rangle\, \Theta(i-\mu_A)\, di\, \langle i|$$

$$\mathbb{M}_A^{\perp} \equiv \mathbb{I} - \mathbb{M}_A = \int_0^1 |i\rangle\, \Theta(\mu_A - i)\, di\, \langle i|$$

<u>Theorem</u>:

If $\mathbb{U} \equiv \int_0^1 |1-i\rangle\, di\, \langle i|$, Then $\mathbb{U}^{-1} = \int_0^1 |i\rangle\, di\, \langle 1-i|$

$$\mathbb{U} \overset{\text{(Discrete)}}{=} \sum_{k=1}^{2N-1|A_j} |2N+1-k\rangle\langle k|$$

<u>Theorem</u>:

$$\mathbb{M}_A^{f} = \mathbb{U}\,\mathbb{M}_A^{\perp}\,\mathbb{U}^{-1}$$

<u>Proof</u>:

$$\begin{aligned}
\mathbb{U}\,\mathbb{M}_A^{\perp}\,\mathbb{U}^{-1} &= \int_0^1\!\int_0^1\!\int_0^1 |1-i\rangle\, di\, \overset{\delta(i-k)}{\cancel{\langle i|k\rangle}}\, \Theta(\mu_A - k)\, dk\, \overset{\delta(k-j)}{\cancel{\langle k|j\rangle}}\, dj\, \langle 1-j| \\
&= \int_0^1 |1-i\rangle\, \Theta(\mu_A - i)\, di\, \langle 1-i| \\
&= \int_0^1 |1-i\rangle \left[1 - \Theta(i-\mu_A)\right] di\, \langle 1-i| \\
\text{Letting } 1-i = j,\quad &= \int_0^1 |j\rangle \left\{1 - \Theta([1-\mu_A] - j)\right\} dj\, \langle j| \\
&= \int_0^1 |j\rangle\, \Theta(j - [1-\mu_A])\, dj\, \langle j| \\
&= \int_0^1 |j\rangle\, \Theta(j - \mu_A^{f})\, dj\, \langle j| \\
&= \mathbb{M}_A^{f}
\end{aligned}$$

Fig. 9: Unitary Transformation from Ortho- to Fuzzy Complement.

In quantum formalisms one considers a type of encodement of patterns and of regularities in physical experience which is at fundamental variance with classical parallel process modeling. The quantum process is described as a "non-selecting measurement." (Schwinger, 1970). It occurs when two empirical processes are possible but there is no empirical proceedure that is capable of distinguishing which one has actually occured. It is thereby empirically meaningless to speak there being individual events (in the classical sense of having a fully defined and fixed sample space). Quantum theory formally provides for this

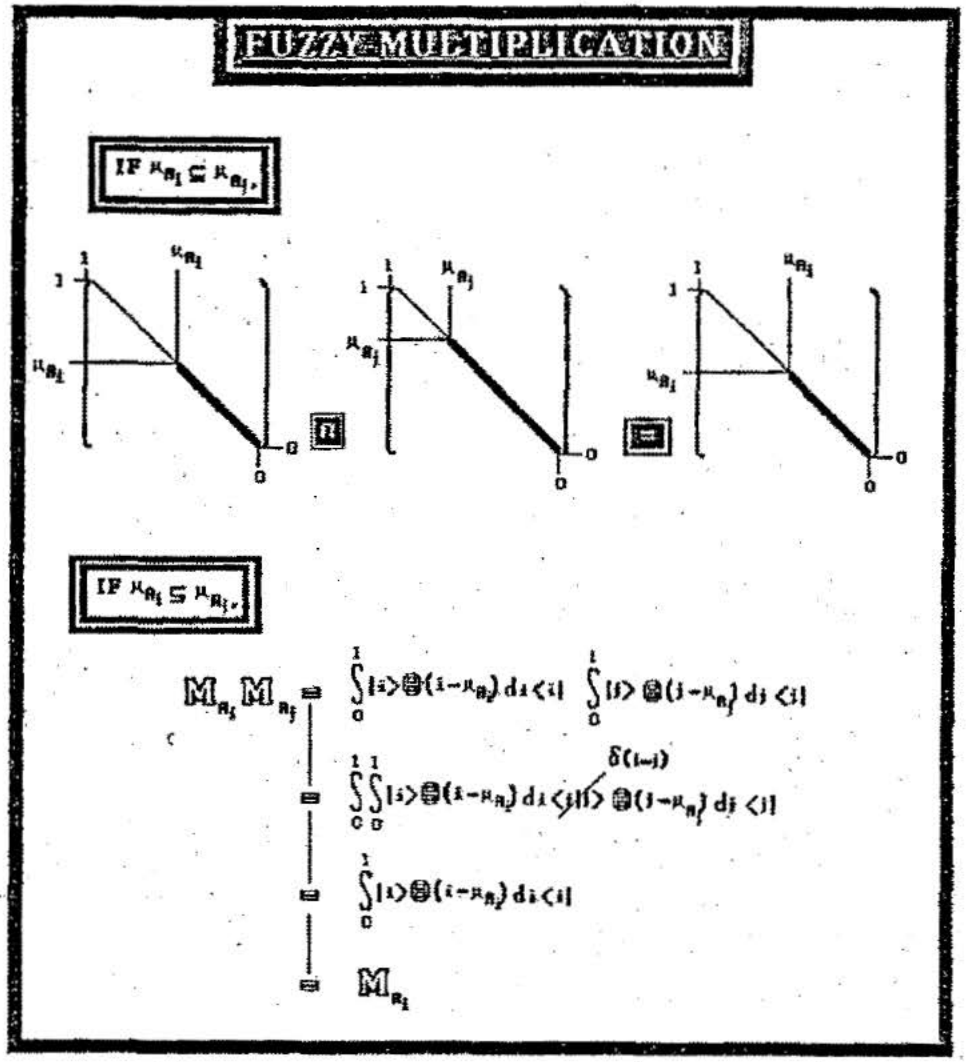


Fig. 10: The Matrix Approach to Fuzzy Multiplication

type of fundamental "ambiguity" by means of symmetries on irreducible equivalence class structures. This can not be formally accomplished within classical models. (4)

As originally discussed by von Weizsacker (Heisenberg, 1958; von Weizsacker, 1955, 1958, 1969, 1971), the lattice of subspaces in Hilbert space provides a meta-logic for our logic. This is true in the following sense. In a non-distributive logic, such as quantum logic, it can be "true" that a statement A is "true or false" without statement A being "true" or statement A being "false."

Over the past several years from his imprisonment in the a Soviet labor camp, Yuri Orlov has been attempting a quantum neurophysiology of concepts that is similar to, although independent of, our own. (Oshins, 1983a) He considers a type of ambiguity which he calls a "doubt state" in a situation that has "inadequate resolution." Orlov's description of Hamlet's soliloquy, "To be or not to be?" (Orlov, 1982) is a good example of the kind of irreducible, non-distributive, possibilistic ambiguity which we have in mind.

One can conceive of a "We" that is "Tom or Jane" but is not "Tom" and is not "Jane." Indeed, although "He or She" may be true, "Black" might be the "adequate resolution" (Orlov, 1981, 1982; Oshins, 1983b) of the experience. Or, one might consider an irreducible concept of "people", as distinct from an enumerative useage, as a sub-collection of attribute filters which are able to be equivocated and thereby ignored. For example, if the construct appropriate to a context asks for "people" then Tom and Jane would be equivocable and gender would not be selected. If the context required a Man/Woman then Tom & Jane would not be equivocated.

We point out that children often form such mathematical "degeneracy". A child frequently will call a cats, dogs, cows, and horses by the same name, say, dog. Our interpretation would be that the child is not selecting out certain percepts or concepts. "Dog" may thus mean "four-legged animal" to the child. Similarly, "Like" is not "Love" and it is not "not-Love," in this sense "Like or not-Like" span all the alternatives. (Oshins and McGoveran, 1979).

The span is the collection of all possible linear superpositions of the vectors being considered. In this sense it is the least upper bound of the possible vectors. Except in the trivial case, it is not an individual vector and it has a higher dimensionality. Fig. 11 illustrates how a simple projective geometry, such as is found in quantum theory, provides a realization of a non-distributive lattice with 3 elements.

We proposed that certain abstract concepts such as "Love/not-Love" & "Like/not-Like" may be complete pairs of linearly dependent alternatives coded with a Hilbert space representation , in that either pair spans a complete space and allows for one to partition ones experience, but that they are linearly dependent within a Hilbert space. The interpretation of Dirac's "principle of linear superposition" (Dirac, 1969; Feynman, 1962) might be found within a Hilbert space superposition of actual firings the neural coding mechanism. (Oshins and McGoveran, Oshins, 1982).

Orlov suggests that quantum logic type of structure might have a role in belief functions or in will functions in terms of ones capacity to choose and implement alternative construct frames of reference. He suggests other cognitive applications and other mental functions which appear to be associated with the superego of psychoanalytic models.

When there is construct ambiguity as to which incompatible frame is appropriate within a quantum formalism, one speaks of indeterminacy between interacting/interfering observables. Such observables are composed of projection operators which are incompatible, ie. they function as "non-collineating filters." Quantum indeterminacy is computed from the trace of commutators, within the matrix algebra. (The commutator of A with respect to B is AB − BA, ie. the difference between the two orderings in matrix multiplication, which vanishes identically in fuzzy logic.)

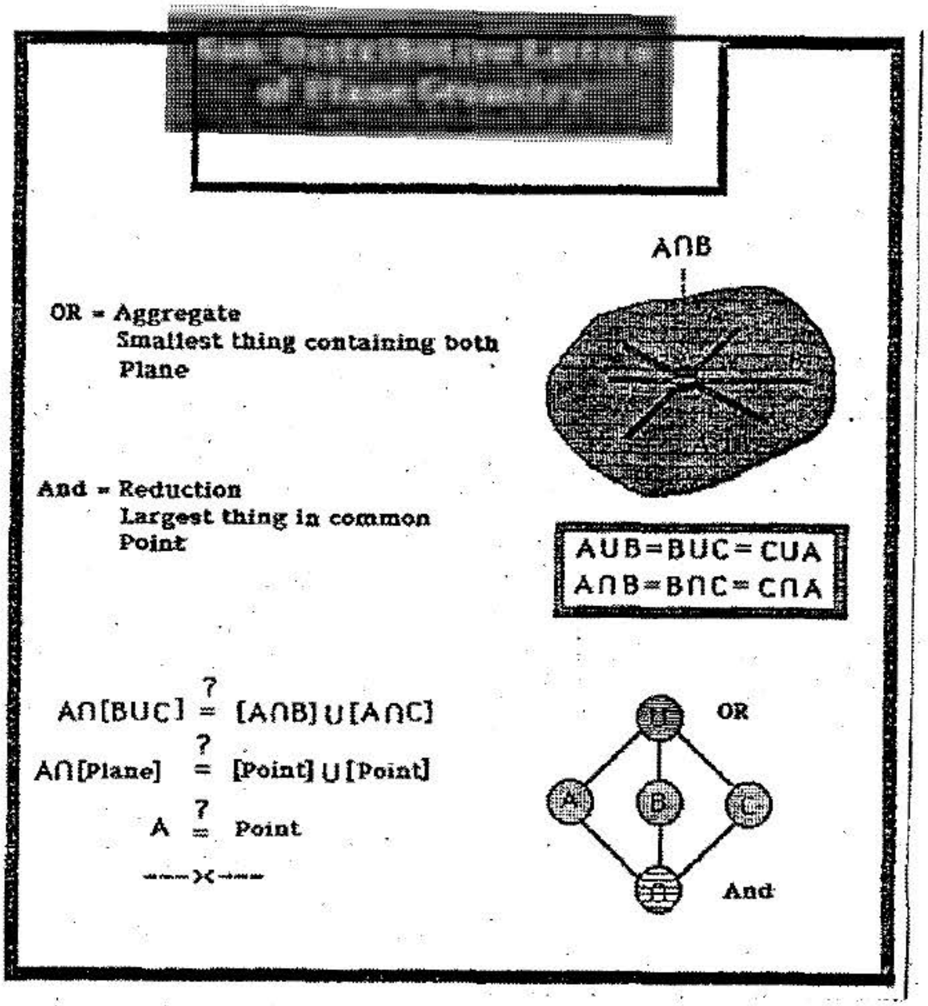


Fig. ii: A Projective Geometry Realization of a Non-Distributive Lattice.

These commutators "generate" change and other transformations within the equivalent but mutually incommensurate (complete ortho-normal) bases systems that "carry the representations." Since the commutators are trivially vacuous for fuzzy membership symbols, they can not provide for such transformations. We have suggested elsewhere (Oshins, 1981) that these equivalence class transformations underly the induced structure of internal mental representations such as have been found by Shepard (Shepard, 1983; Shepard and Chipman, 1970). We would say that the exponentiated, non-commuting algebra of quantum observables "generates" symmetries through the automorphisms of the underlying lattice structure.

Bases whose projection operators do not commute can not be made to aline flush with each other. One can think of this as not being an issue of whether or not the alternatives are Ai or are Aj, where Ai $\longleftrightarrow$ Aj, but whether or not Ai is the case or Bj is the case. That is to say, quantum structures assert strongly that there are such incompatible (ie. it is false that Ai $\longleftrightarrow$ Bj) physical observable structures. These structures compete with eachother for being able to be simultaneously realized. This is the formal meaning of physical indeterminacy.

CONCLUSION

Although we have expressed some reservations in this paper about the efficacy of fuzzy logic to represent the logic of experience, we have found some helpful intuitive understanding from this approach. We believe fuzzy logic to be an interesting and valuable approach to modeling linguistic functions, despite our reservations as to some of the underlying lattice assumptions. Our own approach is similar, but we consider non-distributive lattices.

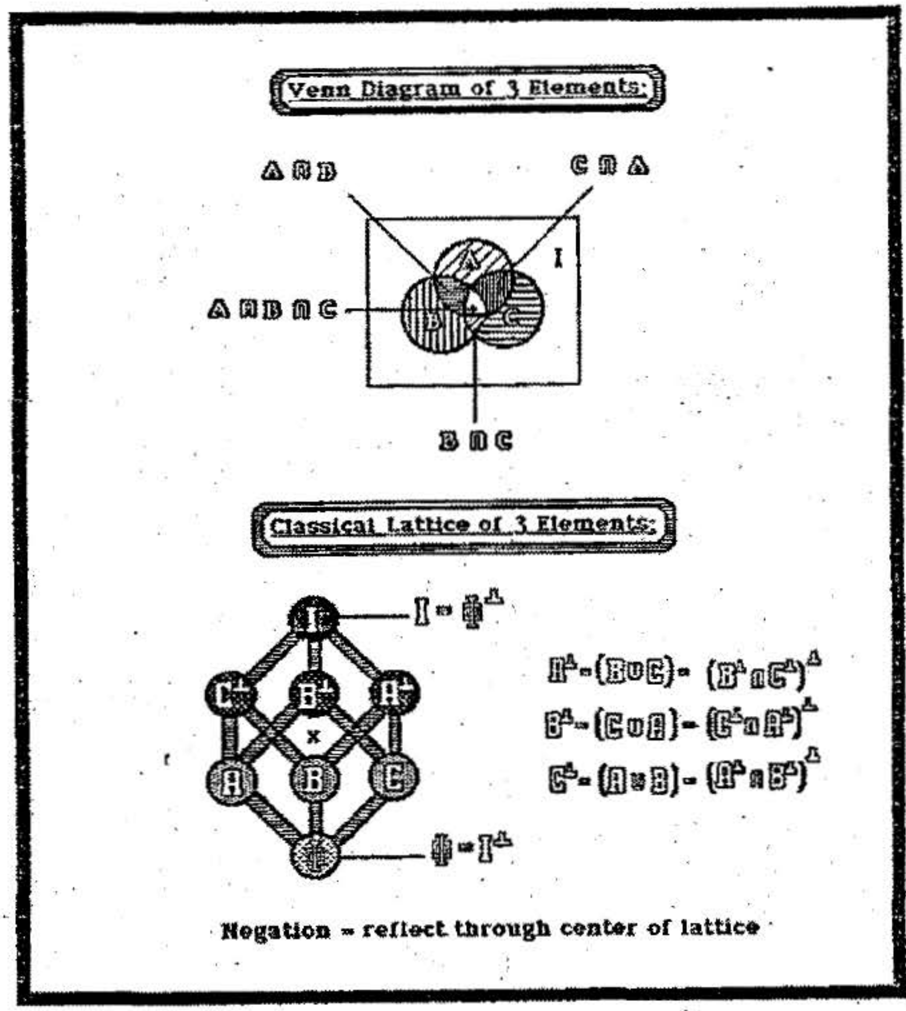
Part of the power of the quantum logic point of view is that there is an isomorphism between the lattice of empirical propositions and the physical formalism. (Birkhofff and von Neumann, 1936; von Neumann, 1955). This provides a natural model to look for neurophysiological corrolates of cognitive/logical structures.

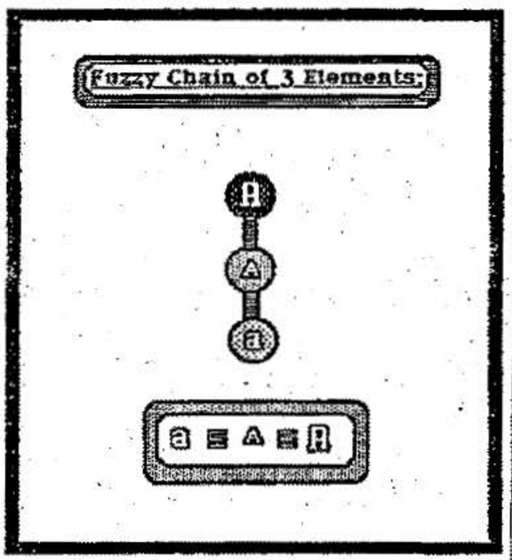
Negationless logics also occur within physical formalisms. (Finkelstein, 1972, 1977) These are brought about from a more inclusive group structure than the unitary one of quantum theory. One can show that in order to have a negative coded in the information content of information signals one must have a synchronization between the preparation of the signal and its measurement. We have suggested elsewhere that such a synchronization process might provide a compactification process that is necessary for the formation of conscious states. (Oshins and McGoveran, 1979; Oshins, 1982). In a future paper, we hope to further address some of the representational and psychological piculiarities of Fuzzy logic.

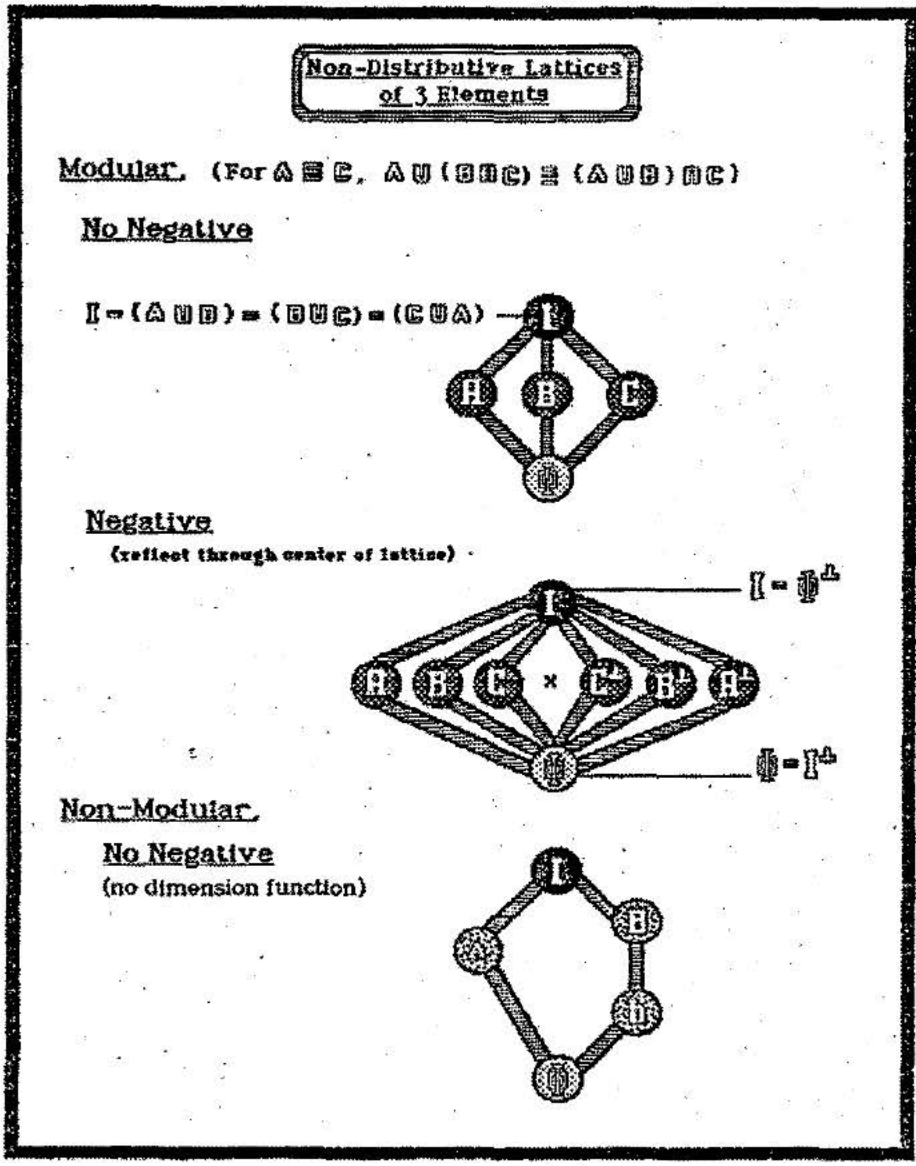
APPENDIX

In this appendix we illustrat the lattice Hasse diagrams for the basic three types of lattices which we have discussed in this paper, the classical, the fuzzy, and the non-distributive. We will consider examples with three elemental building blocks. The rules for drawing the pictures along with the interpretations are found in Figs. 1a-e.

The first diagram is represents the structure of a classical lattice compared with its equivalent Venn Diagram. The second is a fuzzy chain of 3 elements. The last are examples of non-distributive lattices.


Venn Diagram of 3 Elements:
A ∩ B
C ∩ A
I
A ∩ B ∩ C
B ∩ C
Classical Lattice of 3 Elements:
Negation = reflect through center of lattice


Fuzzy Chain of 3 Elements:



It is a threorem that <u>any</u> non-distributive lattice has either the first of the last of these diagrams as a sublattice. In the non-modular case, it seems that one equivocates (forms adjunction with) between the left intermediate level and both of the intermediate levels on the right hand side of the diagram, ie. it equivocates parts with wholes.

All modular lattices have finite "unitary representations" and likewise all finite unitary representations are modular. The weakly-modular lattices have infinite dimensional unitary representations of the symmetries. Should our tentative interpretation that this is how the special linear group in two dimensions would be represented be correct, then the "capacity to modularize" might be necessary for consciousness. It would impose the metric structure needed for a negative to the logic (Finkelstein, 1972, 1977) and make the manifold compact. As mentioned above, we suspect that global neurological synchronization could be responsible.

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FOOTNOTES

1. For A_i, A_j, $A_k \in A$ the relationship $A_i \cup (A_j \cap A_k) \subseteq A_i \cap (A_j \cup A_k)$ holds for any lattice.

2. V. Pinkava (Pinkava, 1971) has introduced a definition of a **well formed classification criterion** (WFCC) as a complete orthonormal basis system. "Every classification criterion which is not a WFCC is called a **spurious classification system** (SCC)." He goes on to suggest that exchanging the truth value of one or more of the basis elements of a WFCC, thereby degrading the classification criterion so that it becomes an SCC, may be responsible for psychotic events such as those exhibited in schizophrenia. Strictly fuzzy is always a SCC within this interpretation.

3. "All diagonal matrices commute, and the product of two diagonal matrices is again diagonal." (Wigner, 1959, p. 8.) As a result, fuzzy logic is only embedded within our matrix algebra which in general would have off-diagonal elements along with the on diagonal ones, ie. there are observable entities whose operators can not be simultaneously diagonalized. One finds such irreducible representations within quantum theories

4. "...there is no difference between the possiility of describing a real or imagined mode of behavior completely and unambiguously in words, and the possibility o realizing it by a finite formal neural network. The two concepts are coextensive." "...every net...can compute only such numbers as can a Turing machine...each of the latter numbers can be computed by such a net...." (von Neumann, 1951).

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